

A New Approach to Scalar Field Theory on Noncommutative Space

Yoshinobu Habara ¹

*Yukawa Institute for Theoretical Physics,
Kyoto University, Kyoto 606-8502, Japan*

Abstract

A new approach to constructing the noncommutative scalar field theory is presented. Not only between \hat{x}_i and \hat{p}_j , we impose commutation relations between \hat{x}_i s as well as \hat{p}_j s, and give a new representation of \hat{x}_i, \hat{p}_j s. We carry out both first- and second-quantization explicitly. The second-quantization is performed in both the operator formalism and the functional integral one.

¹e-mail: habara@yukawa.kyoto-u.ac.jp

1. Introduction

Recently, the extended geometry known as noncommutative geometry is widely discussed in the contexts of field theories, string theories, Matrix models (Ref. [2] [4] [5] [6] etc.). Mathematically (Ref. [1] [3] etc.), the “point” x in noncommutative space is regarded as the operator \hat{x} acting on a Hilbert space and the space coordinates as the eigenvalues of \hat{x} . Until now, in field theories, \ast -product has been used which puts the noncommutativity only to the product between the usual numbers instead of the operators \hat{x} . The field theories following this tactics contains the very interesting problems such as UV/IR mixing (Ref. [7] etc.), while they have divergence of the same degree as the usual field theories. Therefore, using \ast -product as noncommutativity, I believe that one may not be able to obtain consistent quantum gravity model.

In this paper, I would like to propose a new approach to introduce the noncommutativity into the scalar field theory. Our approach is based on the argument that first-quantization is equal to the introduction of noncommutativity into the phase space and that the first-quantized theory to the classical field theory. The quantum mechanics discussed until now have noncommutativity only between the space-time coordinate x_i and the momentum p_j like $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}$, and the representation of this algebra is defined by $\hat{x}_i = x_i$, $\hat{p}_j = -i\frac{\partial}{\partial x_j}$. But I would like to also impose noncommutativity between x_i s as well as p_j s, and make a new representation of \hat{x}_i s and \hat{p}_j s. This new representation leads us to a new Klein-Gordon model with discrete spectrum which can be exactly solved using the harmonic oscillator system in “Euclidean” case. Also, we will carry out second-quantization in both the operator formalism and the functional integral formalism. In the later sections, we will find a slightly strange creation-annihilation algebra of fields and functional integral. This functional integral has special characteristics such as the summation $\sum_{N,l}$ instead of the momentum-integration $\int dk^n$ and the propagator with imaginary part looking like the regulator.

The organization of this paper is as follows. In section 2, we present a noncommutative algebra and its representation. In section 3, we solve Klein-Gordon equation on noncommutative space as first-quantized theory. Section 4 and 5 are devoted to second-quantization by the operator formalism and the functional integral formalism respectively. In section 6, discussions are presented.

2. Representation of Euclidean noncommutative spaces

First of all, we will give an algebra of the “Euclidean” noncommutative \mathbf{R}^n and its conjugate momenta as the first quantized theory.

The commutation relations between the space (or space-time) coordinates \hat{x}_i s are at the order of θ where θ denotes the noncommutativity of the space-time, and those between \hat{x}_i and its conjugate momentum \hat{p}_i are the canonical commutation relations.

$$\begin{aligned}
[\hat{x}_i, \hat{x}_j] &= i\theta_{ij} \\
[\hat{x}_i, \hat{p}_j] &= i\hbar\delta_{ij} \quad (i, j = 1 \sim n) \\
[\hat{p}_i, \hat{p}_j] &= ??
\end{aligned} \tag{1}$$

Though the commutation relations between \hat{p}_i s will be exactly determined later, we shall roughly evaluate them here.

Because of these commutation relations, the deviations Δx and Δp have the following uncertainty relations.

$$\begin{aligned}
[\hat{x}_i, \hat{x}_j] = i\theta_{ij} &\Rightarrow (\Delta x)^2 \geq \theta \\
[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} &\Rightarrow \Delta x \Delta p \geq \hbar, \quad (\Delta x)^2 (\Delta p)^2 \geq \hbar^2
\end{aligned}$$

Thus,

$$[\hat{p}_i, \hat{p}_j] \propto \hbar^2 \theta_{ij}^{-1}$$

In the following discussion, we will take $\hbar = 1$.

Let us describe \hat{x}_i s and \hat{p}_i s directly using the operators in Ref. [2]. Define the operators \hat{U}_i as follows :

$$\begin{aligned}
\hat{U}_i &\equiv e^{i\hat{x}_i} \quad (i, j = 1 \sim n) \\
\hat{U}_i \hat{U}_j &= e^{-i\theta_{ij}} \hat{U}_j \hat{U}_i
\end{aligned} \tag{2}$$

\hat{U}_i s are the operators acting on Hilbert space $\mathcal{H} \ni f(t)$, where $t = (t_1, \dots, t_n) \in \mathbf{R}^n$. The representation of \hat{U}_i s satisfying the commutation relations (2) can be constructed in the following way. We begin by supposing the following form of \hat{U}_i acting on \mathcal{H} :

$$\begin{aligned}
(\hat{U}_i f)(t) &= \chi_i(t) f(t + a_i) \\
\chi_i : \mathbf{R}^n &\rightarrow \mathbf{C} \quad (\text{character homomorphism})
\end{aligned} \tag{3}$$

Using the commutation relations (2), we obtain

$$\begin{aligned}
\hat{U}_i \hat{U}_j f(t) &= \chi_j(a_i) \chi_i(t) \chi_j(t) f(t + a_i + a_j) \\
\hat{U}_j \hat{U}_i f(t) &= \chi_i(a_j) \chi_i(t) \chi_j(t) f(t + a_i + a_j) \\
\text{thus, } e^{-i\theta_{ij}} &= \frac{\chi_j(a_i)}{\chi_i(a_j)}.
\end{aligned} \tag{4}$$

Here we can take

$$a_i = -\theta_{ij}z_j, \quad z_j = (0, \dots, \underset{j}{\widehat{1}}, \dots, 0),$$

$$\chi_i(t) = e^{\frac{i}{2}t_i}.$$

So the representation of U_i s are given by

$$\hat{U}_i f(t) = e^{\frac{i}{2}t_i} f(t - \theta_{ij}z_j). \quad (5)$$

Next, in order to show the representation of \hat{x}_i , we take

$$\begin{aligned} \hat{U}_i &= e^{ik\hat{x}_i} \\ z_i &= (0, \dots, k, \dots, 0) \\ \chi_i(t) &= e^{\frac{i}{2}kt_i}. \end{aligned}$$

Substituting them into (5), we expand in k , and compare the both side at the first order of k to obtain

$$\begin{aligned} e^{ik\hat{x}_1} f(t_1, t_2, \dots, t_n) &= e^{\frac{i}{2}kt_1} f(t_1, t_2 - k\theta_{12}, \dots, t_n - k\theta_{1n}) \\ (1 + ik\hat{x}_1 + \dots) f(t_1, t_2, \dots, t_n) &= (1 + \frac{i}{2}kt_1 + \dots) \{1 - k(\theta_{12}\partial_2 + \dots + \theta_{1n}\partial_n) + \dots\} f(t_1, t_2, \dots, t_n) \\ \Rightarrow \hat{x}_1 &= \frac{1}{2}t_1 + i\theta_{1j}\partial_j, \end{aligned}$$

and thus,

$$\hat{x}_i = \frac{1}{2}t_i + i\theta_{ij}\partial_j. \quad (6)$$

Finally, we determine the representation and the algebra of \hat{p}_i s. In the similar manner to the former construction, we use the operators \hat{U}_i . From the canonical commutation relations $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}$, we have

$$[e^{i\hat{x}_i}, \hat{p}_j] = -e^{i\hat{x}_i} \delta_{ij},$$

thus,

$$\hat{p}_i + \delta_{ij} = \hat{U}_j^{-1} \hat{p}_i \hat{U}_j.$$

If we set $\hat{p}_i f(t) = A_{ij} t_j f(t) + B_{ij} \partial_j f(t)$,

$$\begin{aligned} (\hat{p}_i + \delta_{ij}) f(t) &= (A_{ik} t_k + B_{ik} \partial_k + \delta_{ij}) f(t) \\ \hat{U}_j^{-1} \hat{p}_i \hat{U}_j f(t) &= \{A_{ik}(t_k + \theta_{jl} z_l) + \frac{i}{2} B_{ij} + B_{ik} \partial_k\} f(t). \end{aligned}$$

Thus,

$$\delta_{ij} = A_{ik} \theta_{jk} + \frac{i}{2} B_{ij} = -A_{ik} \theta_{kj} + \frac{i}{2} B_{ij}.$$

Therefore we can choose

$$A_{ij} = -\frac{1}{2} \theta_{ij}^{-1}, \quad B_{ij} = -i \delta_{ij}.$$

So that

$$\begin{aligned} \hat{p}_i &= -\frac{1}{2} \theta_{ij}^{-1} t_j - i \partial_i \\ \Rightarrow [\hat{p}_i, \hat{p}_j] &= -i \theta_{ij}^{-1}. \end{aligned} \tag{7}$$

There may be other choice of A_{ij} and B_{ij} , but the other choices will lead to the different representations and algebras. However, in order to satisfy the relation $(\Delta p)^2 \geq \theta^{-1}$, the variations are restricted.

3. First quantization : deformed Klein-Gordon theory

The purpose of this section is to solve the Klein-Gordon model on the noncommutative space. For simplicity, we will take the dimension of the space to be 2, and the “metric” is taken to be Euclidean.

The Klein-Gordon model, the relativistic first-quantized theory, is constructed from the Einstein’s energy-momentum relation $p_i p_i + m^2 = 0$ by replacing the phase-space coordinates with the operators : $x_i, p_i \rightarrow \hat{x}_i, \hat{p}_i$. On noncommutative space, we use the same method.

We start with the following Klein-Gordon equations :

$$(\hat{p}_i \hat{p}_i + m^2) \phi(t) = 0, \tag{8}$$

$$[(-\frac{1}{2} \theta_{ij}^{-1} t_j - i \partial_i)(-\frac{1}{2} \theta_{ik}^{-1} t_k - i \partial_k) + m^2] \phi(t) = 0. \tag{9}$$

Our space is 2-dimension, so $i, j, k = 1 \sim 2$ and we can take the θ_{ij} 's to be

$$\theta_{ij} = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \theta_{ij}^{-1} = \theta^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (10)$$

Let us describe the operators and equations explicitly in the following :

$$\begin{cases} \hat{x}_1 = \frac{1}{2}t_1 + i\theta\partial_2 \\ \hat{x}_2 = \frac{1}{2}t_2 - i\theta\partial_1 \end{cases}, \quad \begin{cases} \hat{p}_1 = \frac{1}{2\theta}t_2 - i\partial_1 \\ \hat{p}_2 = -\frac{1}{2\theta}t_1 - i\partial_2 \end{cases} \quad (11)$$

$$(\hat{p}_1^2 + \hat{p}_2^2 + m^2)\phi(t_1, t_2) = 0 \quad (12)$$

$$\left[-(\partial_1^2 + \partial_2^2) + \frac{1}{4\theta^2}(t_1^2 + t_2^2) + \frac{i}{\theta}(t_1\partial_2 - t_2\partial_1) + m^2 \right] \phi(t_1, t_2) = 0. \quad (13)$$

The operator part of the Klein-Gordon equation (13) is of the form :

$$\ll \text{2-dimensional harmonic oscillator} \gg + \ll \text{angular momentum} \gg + m^2.$$

The origin of the name “angular momentum” will be appeared later. In this system, the harmonic oscillator is contained. So we introduce the creation-annihilation operators as usual,

$$\begin{cases} \hat{a}_1 = \sqrt{\theta}(\frac{1}{2\theta}t_1 + \partial_1) \\ \hat{a}_1^\dagger = \sqrt{\theta}(\frac{1}{2\theta}t_1 - \partial_1) \end{cases}, \quad \begin{cases} \hat{a}_2 = \sqrt{\theta}(\frac{1}{2\theta}t_2 + \partial_2) \\ \hat{a}_2^\dagger = \sqrt{\theta}(\frac{1}{2\theta}t_2 - \partial_2) \end{cases} \quad (14)$$

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0. \quad (15)$$

Then, $\ll \text{2-dimensional harmonic oscillator} \gg$ part and $\ll \text{angular momentum} \gg$ part become the following form respectively.

$\ll \text{2-dimensional harmonic oscillator} \gg$

$$\begin{aligned} \hat{H}_h &= -(\partial_1^2 + \partial_2^2) + \frac{1}{4\theta^2}(t_1^2 + t_2^2) \\ &= \frac{1}{\theta}(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1) \\ &\equiv \frac{1}{\theta}(\hat{N} + 1) \end{aligned} \quad (16)$$

$\ll \text{angular momentum} \gg$

$$\begin{aligned} \frac{1}{\theta}\hat{L} &\equiv \frac{i}{\theta}(t_1\partial_2 - t_2\partial_1) \\ &= \frac{i}{\theta}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_1 \hat{a}_2^\dagger) \end{aligned} \quad (17)$$

Here, by simple calculation, we see that

$$[\hat{N}, \hat{L}] = 0. \quad (18)$$

Thus, \hat{N} and \hat{L} are simultaneously diagonalizable so that with the eigenvalues (N, l) of (\hat{N}, \hat{L}) , we can write the eigenstates as $|N, l\rangle$.

$$\frac{1}{\theta}(\hat{N} + 1)|N, l\rangle = \frac{1}{\theta}(N + 1)|N, l\rangle \quad (19)$$

$$\frac{i}{\theta}\hat{L}|N, l\rangle = \frac{1}{\theta}l|N, l\rangle \quad (20)$$

From the equation of motion (13), we get the mass-shell condition :

$$\frac{1}{\theta}(N + 1) + \frac{1}{\theta}l + m^2 = 0. \quad (21)$$

Next, we will write down the eigenstates $|N, l\rangle$ explicitly. Since the eigenvalue N is the one of 2-dimensional harmonic oscillator, $|N, l\rangle$ can be decomposed into the two 1-dimensional harmonic oscillators as follows :

$$\begin{aligned} |N, l\rangle &= \sum_{n=0}^N c_n^{(N)}(l) |n, N - n\rangle \\ &= \sum_{n=0}^N c_n^{(N)}(l) |n\rangle \otimes |N - n\rangle. \end{aligned}$$

By substituting this decomposition into (20), we can determine the coefficients $c_n^{(N)}(l)$.

$$\begin{aligned} \hat{L}|N, l\rangle &= (\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_1 \hat{a}_2^\dagger) \sum_{n=0}^N c_n^{(N)}(l) |n, N - n\rangle \\ &= \sum_{n=1}^N c_{n-1}^{(N)}(l) \sqrt{n(N - n + 1)} |n, N - n\rangle \\ &\quad - \sum_{n=0}^{N-1} c_{n+1}^{(N)}(l) \sqrt{(n + 1)(N - n)} |n, N - n\rangle \\ &= -il \sum_{n=0}^N c_n^{(N)}(l) |n, N - n\rangle \\ (n = 0) \quad &-c_1^{(N)}(l) \sqrt{N} = -ilc_0^{(N)}(l) \\ (1 \leq n \leq N - 1) \quad &c_{n-1}^{(N)}(l) \sqrt{n(N - n + 1)} - c_{n+1}^{(N)}(l) \sqrt{(n + 1)(N - n)} = -ilc_n^{(N)}(l) \\ (n = N) \quad &c_{n-1}^{(N)}(l) \sqrt{N} = -ilc_n^{(N)}(l) \end{aligned}$$

All $c_n^{(N)}(l)$ s, ($1 \leq n \leq N$) are described with $c_0^{(N)}(l)$ where $c_0^{(N)}(l)$ is determined by normalizing $|N, l\rangle$.

Now, we shall discuss the eigenvalue l of \hat{L} in detail. We can show immediately the following commutation relations.

$$\begin{cases} [\hat{L}, \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2] = -2i(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger) \\ [\hat{L}, \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger] = 2i(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) \\ [\hat{N}, \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2] = 0 \\ [\hat{N}, \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger] = 0 \end{cases}$$

If we define

$$\hat{L}_\pm \equiv (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger) \pm i(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2), \quad (22)$$

the following commutation relations are derived.

$$[\hat{L}, \hat{L}_\pm] = \pm 2\hat{L}_\pm \quad (23)$$

$$[\hat{N}, \hat{L}_\pm] = 0 \quad (24)$$

Therefore, \hat{L}_\pm raises the eigenvalue of \hat{L} by ± 2 and does not change that of \hat{N} . Furthermore, investigation of the norm of \hat{L}_\pm leads us to determining the range of l .

$$\begin{cases} \hat{L}_+ \hat{L}_- = \hat{N}^2 - \hat{L}^2 + 2\hat{N} + 2\hat{L} \\ \hat{L}_- \hat{L}_+ = \hat{N}^2 - \hat{L}^2 + 2\hat{N} - 2\hat{L} \end{cases}$$

$$\langle N, l | \hat{L}_\pm \hat{L}_\mp | N, l \rangle = (N \pm l)(N \mp l + 2) \geq 0$$

$$\text{thus, } -N \leq l \leq N \quad \text{or} \quad l = -N, -N+2, -N+4, \dots, N-4, N-2, N$$

This is because we call $\hat{L} \ll \text{angular momentum} \gg$.

If we take $\theta m^2 = -1$, there exists on-shell state $l = -N$ for all N .

At last, since we have obtained the complete system, we can decompose the field ϕ into its component fields in the following way.

$$\begin{aligned} \phi(t_1, t_2) &= \sum_{N=0}^{\infty} \sum_{-N \leq l \leq N} \alpha_{n,l} \phi_{N,l}(t_1, t_2) \\ &= \sum_{N=0}^{\infty} \sum_{-N \leq l \leq N} \alpha_{n,l} \langle t_1, t_2 | N, l \rangle \end{aligned} \quad (25)$$

Now, to finish the discussion on field ϕ , we argue the field ϕ^\dagger . The method is very similar to that of ϕ , we only sketch the main results.

$$\phi^\dagger(t_1, t_2) = \sum_{N=0}^{\infty} \sum_{-N \leq l \leq N} \alpha_{N,l}^\dagger \langle N, l | t_1, t_2 \rangle \quad (26)$$

equation of motion

$$\left[-(\partial_1^2 + \partial_2^2) + \frac{1}{4\theta^2}(t_1^2 + t_2^2) - \frac{i}{\theta}(t_1\partial_2 - t_2\partial_1) + m^2 \right] \phi^\dagger(t_1, t_2) = 0 \quad (27)$$

mass-shell condition

$$\frac{1}{\theta}(N+1) - \frac{1}{\theta}l + m^2 = 0 \quad (28)$$

Although the usual scalar field theory on the commutative space has the continuous spectrum, we find that the above model has the discrete spectrum (N, l) . This is the desirable result for the noncommutative space quantum mechanics. Because of the uncertainty of the space-time itself (and also the momentum), the spectrum may get the discreteness.

So far we have discussed the “Euclidean” space, but if we consider in the “Minkovski” space such that $-\hat{p}_1^2 + \hat{p}_2^2 + m^2 = 0$, we can’t first-quantize by using harmonic oscillator system. The “Minkovski” space will be discussed in future works.

4. Second quantization : scalar field theory on noncommutative space

To begin this section, let us write down the action which produces the equation of motion (13),

$$S = \int dt_1 dt_2 \left[\partial_i \phi^\dagger \partial_i \phi + \frac{1}{4\theta^2} t_i t_i \phi^\dagger \phi + \frac{i}{2\theta} \{ \phi^\dagger \epsilon_{ij} t_i \partial_j \phi - \phi \epsilon_{ij} t_i \partial_j \phi^\dagger \} + m^2 \phi^\dagger \phi \right] \quad (29)$$

with $\epsilon_{12} = 1$. To second-quantize the fields ϕ and ϕ^\dagger , we must define the conjugate momenta π and π^\dagger . Usually, to define momentum we use time-coordinate, but in this “Euclidean” space, there are no special difference between t_1, t_2 , and thus we define four momenta $\pi_1, \pi_2, \pi_1^\dagger, \pi_2^\dagger$ as follows.

$$\begin{cases} \pi_1 \equiv \frac{\delta S}{\delta(\partial_1 \phi)} = (\partial_1 - \frac{i}{2\theta} t_2) \phi^\dagger \\ \pi_2 \equiv \frac{\delta S}{\delta(\partial_2 \phi)} = (\partial_2 + \frac{i}{2\theta} t_1) \phi^\dagger \end{cases} \quad (30)$$

$$\begin{cases} \pi_1^\dagger \equiv \frac{\delta S}{\delta(\partial_1 \phi^\dagger)} = (\partial_1 + \frac{i}{2\theta} t_2) \phi \\ \pi_2^\dagger \equiv \frac{\delta S}{\delta(\partial_2 \phi^\dagger)} = (\partial_2 - \frac{i}{2\theta} t_1) \phi \end{cases} \quad (31)$$

For the moment, we shall closely look at the operators in (29) and (30). This is needed for getting the algebra of the coefficients $\alpha_{N,l}$, $\alpha_{N,l}^\dagger$. The following commutation relations can be easily checked :

$$\begin{cases} [\hat{L}, (\partial_1 - \frac{i}{2\theta} t_2)] = -i(\partial_2 + \frac{i}{2\theta} t_1) \\ [\hat{L}, (\partial_2 + \frac{i}{2\theta} t_1)] = i(\partial_1 - \frac{i}{2\theta} t_2) \end{cases}, \quad \begin{cases} [\hat{L}, (\partial_1 + \frac{i}{2\theta} t_2)] = -i(\partial_2 - \frac{i}{2\theta} t_1) \\ [\hat{L}, (\partial_2 - \frac{i}{2\theta} t_1)] = i(\partial_1 + \frac{i}{2\theta} t_2) \end{cases} \quad (32)$$

$$\begin{cases} [\hat{N}, (\partial_1 - \frac{i}{2\theta} t_2)] = i(\partial_2 + \frac{i}{2\theta} t_1) \\ [\hat{N}, (\partial_2 + \frac{i}{2\theta} t_1)] = -i(\partial_1 - \frac{i}{2\theta} t_2) \end{cases}, \quad \begin{cases} [\hat{N}, (\partial_1 + \frac{i}{2\theta} t_2)] = -i(\partial_2 - \frac{i}{2\theta} t_1) \\ [\hat{N}, (\partial_2 - \frac{i}{2\theta} t_1)] = i(\partial_1 + \frac{i}{2\theta} t_2) \end{cases} \quad (33)$$

Let us define the “raising and lowering” operators :

$$\hat{N}_\pm \equiv (\partial_1 - \frac{i}{2\theta} t_2) \pm i(\partial_2 + \frac{i}{2\theta} t_1) \quad (34)$$

$$\hat{N}_\pm^\dagger \equiv (\partial_1 + \frac{i}{2\theta} t_2) \mp i(\partial_2 - \frac{i}{2\theta} t_1). \quad (35)$$

$$\begin{cases} [\hat{N}, \hat{N}_\pm] = \pm \hat{N}_\pm \\ [\hat{L}, \hat{N}_\pm] = \mp \hat{N}_\pm \end{cases}, \quad \begin{cases} [\hat{N}, \hat{N}_\pm^\dagger] = \pm \hat{N}_\pm^\dagger \\ [\hat{L}, \hat{N}_\pm^\dagger] = \pm \hat{N}_\pm^\dagger \end{cases} \quad (36)$$

Consequently, \hat{N}_\pm raises the eigenvalues of \hat{N} and \hat{L} by ± 1 and ∓ 1 respectively, and \hat{N}_\pm^\dagger does so to \hat{N}^\dagger , \hat{L}^\dagger by ± 1 , ± 1 . So, if ϕ and ϕ^\dagger are acted on by \hat{N}_\pm and \hat{N}_\pm^\dagger respectively, they remain in the on-shell states, in other words, keep the mass-shell conditions (21)(28).

Next, we shall investigate the norms of \hat{N}_\pm and \hat{N}_\pm^\dagger . This can be done by the following equations :

$$\begin{cases} \hat{N}_\pm \hat{N}_\mp = -\frac{1}{\theta}(\hat{N} + 1) + \frac{1}{\theta}\hat{L} \pm \frac{1}{\theta} \\ \langle N, l | \hat{N}_\pm \hat{N}_\mp | N, l \rangle = -\frac{1}{\theta}\{(N + 1) - l \mp 1\} \\ \hat{N}_\pm | N, L \rangle = -\frac{i}{\sqrt{\theta}}\sqrt{(N + 1) - l \pm 1} | N \pm 1, l \mp 1 \rangle, \end{cases} \quad (37)$$

$$\begin{cases} \hat{N}_\pm^\dagger \hat{N}_\mp^\dagger = -\frac{1}{\theta}(\hat{N} + 1) - \frac{1}{\theta}\hat{L} \pm \frac{1}{\theta} \\ \langle N, l | \hat{N}_\pm^\dagger \hat{N}_\mp^\dagger | N, l \rangle = -\frac{1}{\theta}\{(N + 1) + l \mp 1\} \\ \hat{N}_\pm^\dagger | N, L \rangle = -\frac{i}{\sqrt{\theta}}\sqrt{(N + 1) + l \pm 1} | N \pm 1, l \pm 1 \rangle. \end{cases} \quad (38)$$

Now that we have understood the operators appearing in the momenta $\pi_1, \pi_2, \pi_1^\dagger, \pi_2^\dagger$, let us quantize the fields ϕ, ϕ^\dagger . The procedure is familiar one and gives the “equal-time” commutation relations between the fields ϕ, ϕ^\dagger and its conjugate momenta $\pi_1, \pi_2, \pi_1^\dagger, \pi_2^\dagger$.

$$\begin{cases} [\hat{\phi}(t_1, t_2), \hat{\pi}_1(t_1, t'_2)] = i\delta(t_2 - t'_2) \\ [\hat{\phi}(t_1, t_2), \hat{\pi}_2(t'_1, t_2)] = i\delta(t_1 - t'_1) \end{cases} \quad (39)$$

$$\begin{cases} [\hat{\phi}^\dagger(t_1, t_2), \hat{\pi}_1^\dagger(t_1, t'_2)] = i\delta(t_2 - t'_2) \\ [\hat{\phi}^\dagger(t_1, t_2), \hat{\pi}_2^\dagger(t'_1, t_2)] = i\delta(t_1 - t'_1) \end{cases} \quad (40)$$

These are all the Schrödinger operators.

By acting the following integration operators on each equations such that

$$\begin{aligned} & \int dt_2 \int dt'_2 \langle N, l | t_1, t_2 \rangle \langle t_1, t'_2 | N', l' \rangle \quad \text{acts on the first equation of (39),} \\ & \int dt_1 \int dt'_1 \langle N, l | t_1, t_2 \rangle \langle t'_1, t_2 | N', l' \rangle \quad \text{acts on the second equation of (39),} \\ & \int dt_2 \int dt'_2 \langle t_1, t_2 | N', l' \rangle \langle N, l | t_1, t'_2 \rangle \quad \text{acts on the first equation of (40),} \\ & \int dt_1 \int dt'_1 \langle t_1, t_2 | N', l' \rangle \langle N, l | t'_1, t_2 \rangle \quad \text{acts on the second equation of (40),} \end{aligned}$$

and using the completeness conditions : $\int dt_2 | t_1, t_2 \rangle \langle t_1, t_2 | = 1$, $\int dt_1 | t_1, t_2 \rangle \langle t_1, t_2 | = 1$, we obtain the commutation relations about the coefficients $\alpha_{N,l}, \alpha_{N,l}^\dagger$.

$$\left[\hat{\alpha}_{N,l}, \sum_{N'',l''} \hat{\alpha}_{N'',l''}^\dagger (\partial_1 - \frac{i}{2\theta} t_2) \langle N'', l'' | N', l' \rangle \right] = i \delta_{NN'} \delta_{ll'} \quad (41)$$

$$\left[\hat{\alpha}_{N,l}, \sum_{N'',l''} \hat{\alpha}_{N'',l''}^\dagger (\partial_2 + \frac{i}{2\theta} t_1) \langle N'', l'' | N', l' \rangle \right] = i \delta_{NN'} \delta_{ll'} \quad (42)$$

$$\left[\hat{\alpha}_{N,l}^\dagger, \sum_{N'',l''} \hat{\alpha}_{N'',l''} \langle N', l' | (\partial_1 + \frac{i}{2\theta} t_2) | N'', l'' \rangle \right] = i \delta_{NN'} \delta_{ll'} \quad (43)$$

$$\left[\hat{\alpha}_{N,l}^\dagger, \sum_{N'',l''} \hat{\alpha}_{N'',l''} \langle N', l' | (\partial_2 - \frac{i}{2\theta} t_1) | N'', l'' \rangle \right] = i \delta_{NN'} \delta_{ll'} \quad (44)$$

By making (41) $\pm i \times$ (42), and (43) $\mp i \times$ (44), we obtain

$$\left[\hat{\alpha}_{N,l}, \sum_{N'',l''} \hat{\alpha}_{N'',l''}^\dagger \hat{N}_\pm \langle N'', l'' | N', l' \rangle \right] = i \delta_{NN'} \delta_{ll'} \quad (45)$$

$$\left[\hat{\alpha}_{N,l}^\dagger, \sum_{N'',l''} \hat{\alpha}_{N'',l''} \langle N', l' | \hat{N}_\pm^\dagger | N'', l'' \rangle \right] = i \delta_{NN'} \delta_{ll'}. \quad (46)$$

From (45)(46), we find that

$$[\hat{\alpha}_{N,l}, \hat{\alpha}_{N',\pm 1, l' \mp 1}^\dagger] = (1 \pm i) \frac{\sqrt{\theta}}{\sqrt{(N+1) - l \pm 1}} \delta_{NN'} \delta_{ll'} \quad (47)$$

$$[\hat{\alpha}_{N,l}^\dagger, \hat{\alpha}_{N',\pm 1, l' \pm 1}] = (1 \mp i) \frac{\sqrt{\theta}}{\sqrt{(N+1) + l \pm 1}} \delta_{NN'} \delta_{ll'}. \quad (48)$$

From the suffixes of $\hat{\alpha}, \hat{\alpha}^\dagger$ in (47)(48), we find that they satisfy the mass-shell conditions of ϕ, ϕ^\dagger for each (47), (48). So, these are the algebras which create-annihilate the field ϕ, ϕ^\dagger respectively. But the algebras are fairly complex, so we may leave the explicit construction of the complete Fock space to future works.

5. Second quantization : functional integral

In this section, we will construct the functional integral formalism of the second quantization of the scalar field theory on noncommutative space. According to Ref. [8], we start with the “Minkovskian” version of the action (29), and use the “imaginary-time” evolution operator in order to define the “Euclidean” functional integral. Although our method is very similar to that of usual scalar field theories, I would like to follow the calculation carefully, because our result is slightly different.

The “Minkovslian” Klein-Gordon model is defined in the following way.

$$\begin{cases} \hat{x}_1 = \frac{1}{2}t_1 + i\theta\partial_2 \\ \hat{x}_2 = \frac{1}{2}t_2 - i\theta\partial_1 \end{cases}, \quad \begin{cases} \hat{p}_1 = -\frac{1}{2\theta}t_2 + i\partial_1 \\ \hat{p}_2 = -\frac{1}{2\theta}t_1 - i\partial_2 \end{cases} \quad (49)$$

$$(-\hat{p}_1^2 + \hat{p}_2^2 + m^2)\phi(t_1, t_2) = 0 \quad (50)$$

$$\left[(\partial_1^2 - \partial_2^2) + \frac{1}{4\theta^2}(t_1^2 - t_2^2) + \frac{i}{\theta}(t_1\partial_2 + t_2\partial_1) + m^2 \right] \phi(t_1, t_2) = 0 \quad (51)$$

$$S_M = \int dt_1 dt_2 \left[\partial_1 \phi^\dagger \partial_1 \phi - \partial_2 \phi^\dagger \partial_2 \phi - \frac{1}{4\theta^2}(t_1^2 - t_2^2) \phi^\dagger \phi - \frac{i}{2\theta}(\phi^\dagger t_1 \partial_2 \phi + \phi^\dagger t_2 \partial_1 \phi - \phi t_1 \partial_2 \phi^\dagger - \phi t_2 \partial_1 \phi^\dagger) - m^2 \phi^\dagger \phi \right] \quad (52)$$

$$\begin{cases} \pi_1 \equiv \frac{\delta S}{\delta(\partial_1 \phi)} = (\partial_1 - \frac{i}{2\theta}t_2)\phi^\dagger \\ \pi_1^\dagger \equiv \frac{\delta S}{\delta(\partial_1 \phi^\dagger)} = (\partial_1 + \frac{i}{2\theta}t_2)\phi \end{cases} \quad (53)$$

The definition of the Hamiltonian is supposed to be the same as the usual one.

$$\begin{aligned} \mathfrak{H} &\equiv \pi_1 \partial_1 \phi + \pi_1^\dagger \partial_1 \phi^\dagger - \mathfrak{L}_M \\ &= \partial_1 \phi^\dagger \partial_1 \phi + \partial_2 \phi^\dagger \partial_2 \phi + \frac{1}{4\theta^2}(t_1^2 - t_2^2) \phi^\dagger \phi - \frac{i}{2\theta}(\phi^\dagger t_1 \partial_2 \phi - \phi t_1 \partial_2 \phi^\dagger) - m^2 \phi^\dagger \phi \\ &= (\pi_1^\dagger - \frac{i}{2\theta}t_2 \phi)(\pi_1 + \frac{i}{2\theta}t_2 \phi^\dagger) + \partial_2 \phi^\dagger \partial_2 \phi + \frac{1}{4\theta^2}(t_1^2 - t_2^2) \phi^\dagger \phi \\ &\quad - \frac{i}{2\theta}(\phi^\dagger t_1 \partial_2 \phi - \phi t_1 \partial_2 \phi^\dagger) - m^2 \phi^\dagger \phi \end{aligned} \quad (54)$$

$$H(t_1) \equiv \int dt_2 \mathfrak{H} \equiv \int dt_2 (\pi_1^\dagger - \frac{i}{2\theta}t_2 \phi)(\pi_1 + \frac{i}{2\theta}t_2 \phi^\dagger) + V[\phi, \phi^\dagger; t_1] \quad (55)$$

Then, we shall impose the canonical commutation relations.

$$\begin{cases} [\hat{\phi}(t_1, t_2), \hat{\pi}_1(t_1, t'_2)] = i\delta(t_2 - t'_2) \\ [\hat{\phi}^\dagger(t_1, t_2), \hat{\pi}_1^\dagger(t_1, t'_2)] = i\delta(t_2 - t'_2) \end{cases} \quad (56)$$

$$\begin{cases} \hat{\pi}_1(t_1, t_2) = -i \frac{\delta}{\delta \phi(t_1, t_2)} \\ \hat{\pi}_1^\dagger(t_1, t_2) = -i \frac{\delta}{\delta \phi^\dagger(t_1, t_2)} \end{cases} : \text{Schrödinger operator} \quad (57)$$

Suppose that $\hat{\phi}_H(t_1, t_2)$ and $\hat{\phi}_H^\dagger(t_1, t_2)$ are Heisenberg operators at “time” t_1 . In Heisenberg representation, we have

$$\begin{aligned}\hat{\phi}_H(t_1, t_2)|\phi(t_1, t_2); t_1\rangle_H &= \phi|\phi(t_1, t_2); t_1\rangle_H \\ \hat{\phi}_H^\dagger(t_1, t_2)|\phi(t_1, t_2); t_1\rangle_H &= \phi^\dagger|\phi(t_1, t_2); t_1\rangle_H \\ \text{completeness} \quad \int \mathfrak{D}\phi \mathfrak{D}\phi^\dagger |\phi(t_1, t_2); t_1\rangle_{HH} \langle\phi(t_1, t_2); t_1| &= 1 \\ \text{orthonormality} \quad {}_H\langle\phi(t_1, t_2); t_1|\phi(t_1, t'_2); t_1\rangle_H &= \delta^2[\phi(t_1, t_2) - \phi(t_1, t'_2)].\end{aligned}$$

Then, let us investigate the transition amplitude from “time” $t_{1,i}$ to $t_{1,f}$, and cut the interval $(t_{1,f} - t_{1,i})$ into a small piece $\epsilon \ll 1$, where $(t_{1,f} - t_{1,i}) = (N + 1)\epsilon$, $N \gg 1$.

$$\begin{aligned}{}_H\langle\phi_f(t_{1,f}, t_{2,f}); t_{1,f}|\phi_i(t_{1,i}, t_{2,i}); t_{1,i}\rangle_H \\ = \int \mathfrak{D}\phi_N \mathfrak{D}\phi_N^\dagger \cdots \int \mathfrak{D}\phi_1 \mathfrak{D}\phi_1^\dagger \\ \times {}_H\langle\phi_f(t_{1,f}, t_{2,f}); t_{1,f}|\phi_N(t_{1,N}, t_{2,N}); t_{1,N}\rangle_{HH} \langle\phi_N(t_{1,N}, t_{2,N}); t_{1,N}| \cdots \\ \cdots |\phi_1(t_{1,1}, t_{2,1}); t_{1,1}\rangle_{HH} \langle\phi_1(t_{1,1}, t_{2,1}); t_{1,1}|\phi_i(t_{1,i}, t_{2,i}); t_{1,i}\rangle_H\end{aligned}\quad (58)$$

Now, we shall go to the Schrödinger representation. Let us express the evolution operator as $\hat{U}(t_1, t'_1)$. We obtain

$$\begin{aligned}\text{Schrödinger eq.} \quad -\partial_1 \hat{U}(t_1, t'_1) &= \hat{H}(t_1) \hat{U}(t_1, t'_1) \\ \text{for } t_1 \geq t'_1, \quad \hat{U}(t_1, t_1) &= 1.\end{aligned}\quad (59)$$

Since the Hamiltonian manifestly depends on the “time” t_1 , $\hat{U}(t_1, t'_1) \neq e^{-\hat{H}(t_1 - t'_1)}$.

Next, we must evaluate the transition amplitude during the small interval $(t_{1,k} - t_{1,k-1}) = \epsilon$ by using the Schrödinger representation $|\phi(t_1, t_2)\rangle_S$.

$$\begin{aligned}{}_H\langle\phi_k(t_{1,k}, t_{2,k}); t_{1,k}|\phi_{k-1}(t_{1,k-1}, t_{2,k-1}); t_{1,k-1}\rangle_H \\ = {}_S\langle\phi_k(t_{1,k}, t_{2,k})|\hat{U}(t_{1,k}, t_{1,k-1})|\phi_{k-1}(t_{1,k-1}, t_{2,k-1})\rangle_S\end{aligned}\quad (60)$$

Substituting (60) into the Schrödinger equation (59), we have

$$\begin{aligned}-\partial_{1,k} {}_S\langle\phi_k(t_{1,k}, t_{2,k})|\hat{U}(t_{1,k}, t_{1,k-1})|\phi_{k-1}(t_{1,k-1}, t_{2,k-1})\rangle_S \\ = \hat{H}(t_{1,k}) {}_S\langle\phi_k(t_{1,k}, t_{2,k})|\hat{U}(t_{1,k}, t_{1,k-1})|\phi_{k-1}(t_{1,k-1}, t_{2,k-1})\rangle_S.\end{aligned}\quad (61)$$

Here, in replacing the classical Hamiltonian $H(t_1)$ by the quantum Hamiltonian $\hat{H}(t_1)$, the ambiguity of the operator ordering arises in the terms such as $\pi_1\phi$, $\pi_1^\dagger\phi^\dagger$. So, we define the quantum Hamiltonian by using the Weyl ordering. For example,

$$\begin{aligned}
& \int_{t_{2,k-1}}^{t_{2,k}} dt_2 \pi_1 \phi \\
& \rightarrow \int_{t_{2,k-1}}^{t_{2,k}} dt_2 \{ \hat{\pi}_1 \hat{\phi} \}_W \\
& \equiv \frac{1}{2} \int_{t_{2,k-1}}^{t_{2,k}} dt_2 \{ \hat{\pi}_{1,k}(t_{1,k}, t_2) \hat{\phi}_{k-1}(t_{1,k-1}, t_2) + \hat{\phi}_k(t_{1,k}, t_2) \hat{\pi}_{1,k-1}(t_{1,k-1}, t_2) \}. \quad (62)
\end{aligned}$$

Then,

$$\begin{aligned}
& \int_{t_{2,k-1}}^{t_{2,k}} dt_2 \{ \hat{\pi}_1 \hat{\phi} \}_W S \langle \phi_k(t_{1,k}, t_{2,k}) | \hat{U}(t_{1,k}, t_{1,k-1}) | \phi_{k-1}(t_{1,k-1}, t_{2,k-1}) \rangle_S \\
& = \frac{1}{2} \int_{t_{2,k-1}}^{t_{2,k}} dt_2 \left\{ -i \frac{\delta}{\delta \phi_k(t_{1,k}, t_2)} \phi_{k-1}(t_{1,k-1}, t_2) + \phi_k(t_{1,k}, t_2) \cdot i \frac{\delta}{\delta \phi_{k-1}(t_{1,k-1}, t_2)} \right\} \\
& \quad \times_S \langle \phi_k(t_{1,k}, t_{2,k}) | \hat{U}(t_{1,k}, t_{1,k-1}) | \phi_{k-1}(t_{1,k-1}, t_{2,k-1}) \rangle_S. \quad (63)
\end{aligned}$$

The same thing can be said about $\pi_1^\dagger \phi^\dagger$. To solve the Schrödinger equation (61), let us define

$$\begin{aligned}
& S \langle \phi_k(t_{1,k}, t_{2,k}) | \hat{U}(t_{1,k}, t_{1,k-1}) | \phi_{k-1}(t_{1,k-1}, t_{2,k-1}) \rangle_S \\
& \equiv \int \mathfrak{D}\pi_{1,k} \mathfrak{D}\pi_{1,k}^\dagger u_{\pi_{1,k}}(\epsilon) \\
& \quad \times \exp \left[i \int_{t_{2,k-1}}^{t_{2,k}} dt_2 \pi_{1,k}(t_{1,k}, t_2) \left\{ \phi_k(t_{1,k}, t_2) - \phi_{k-1}(t_{1,k-1}, t_2) \right\} \right. \\
& \quad \left. + i \int_{t_{2,k-1}}^{t_{2,k}} dt_2 \pi_{1,k}^\dagger(t_{1,k}, t_2) \left\{ \phi_k^\dagger(t_{1,k}, t_2) - \phi_{k-1}^\dagger(t_{1,k-1}, t_2) \right\} \right]. \quad (64)
\end{aligned}$$

boundary condition : $u_{\pi_{1,k}}(0) = 1$

Substituting (64) into (63) with omitting the (t_1, t_2) -dependence, we obtain

$$\begin{aligned}
& \{ \hat{\pi}_1 \hat{\phi} \}_W S \langle \phi_k | \hat{U}(\epsilon) | \phi_{k-1} \rangle_S \\
& = \left\{ \left(-i \frac{\delta}{\delta \phi_k} \right) \phi_{k-1} + \phi_k \left(i \frac{\delta}{\delta \phi_{k-1}} \right) \right\} \\
& \quad \times \int \mathfrak{D}\pi_{1,k} \mathfrak{D}\pi_{1,k}^\dagger u_{\pi_{1,k}}(\epsilon) \exp \left[i \pi_{1,k}(\phi_k - \phi_{k-1}) + i \pi_{1,k}^\dagger(\phi_k^\dagger - \phi_{k-1}^\dagger) \right] \\
& = \int \mathfrak{D}\pi_{1,k} \mathfrak{D}\pi_{1,k}^\dagger \pi_{1,k} \frac{\phi_k + \phi_{k-1}}{2} u_{\pi_{1,k}}(\epsilon) \exp \left[i \pi_{1,k}(\phi_k - \phi_{k-1}) + i \pi_{1,k}^\dagger(\phi_k^\dagger - \phi_{k-1}^\dagger) \right]
\end{aligned}$$

The same for $\pi_1^\dagger \phi^\dagger$.

At last, let us solve the Schödinger equation (61). The equation is

$$\begin{aligned}
& -\partial_\epsilon {}_S\langle\phi_k|\hat{U}(\epsilon)|\phi_{k-1}\rangle_S \\
& = \hat{H}(t_{1,k})_S\langle\phi_k|\hat{U}(\epsilon)|\phi_{k-1}\rangle_S \\
& = \left[\left\{ (\hat{\pi}_1^\dagger - \frac{i}{2\theta}t_2\hat{\phi})(\hat{\pi}_1 + \frac{i}{2\theta}t_2\hat{\phi}^\dagger) \right\}_W + V[\hat{\phi}, \hat{\phi}^\dagger; t_{1,k}] \right]_S \langle\phi_k|\hat{U}(\epsilon)|\phi_{k-1}\rangle_S.
\end{aligned}$$

For $u_{\pi_{1,k}}(\epsilon)$,

$$\begin{aligned}
& -\partial_\epsilon u_{\pi_{1,k}}(\epsilon) \\
& = \left\{ \pi_{1,k}^\dagger \pi_{1,k} - \frac{i}{2\theta}t_2 \frac{\phi_k + \phi_{k-1}}{2} \pi_{1,k} + \frac{i}{2\theta}t_2 \frac{\phi_k^\dagger + \phi_{k-1}^\dagger}{2} \pi_{1,k}^\dagger + \frac{1}{4\theta^2}t_2^2 \phi_k^\dagger \phi_k \right. \\
& \quad \left. + V[\phi, \phi^\dagger; t_{1,k-1}] + \frac{\partial V[\phi_k, \phi_k^\dagger; t_{1,k}]}{\partial t_{1,k}} \Big|_{t_{1,k}=t_{1,k-1}} \epsilon + \dots \right\} u_{\pi_{1,k}}(\epsilon).
\end{aligned}$$

Thus, the solution is

$$\begin{aligned}
u_{\pi_{1,k}}(\epsilon) = \exp \Big[-\epsilon \Big\{ \pi_{1,k}^\dagger \pi_{1,k} - \frac{i}{2\theta}t_2 \frac{\phi_k + \phi_{k-1}}{2} \pi_{1,k} + \frac{i}{2\theta}t_2 \frac{\phi_k^\dagger + \phi_{k-1}^\dagger}{2} \pi_{1,k}^\dagger \\
+ \frac{1}{4\theta^2}t_2^2 \phi_k^\dagger \phi_k + V[\phi, \phi^\dagger; t_{1,k-1}] \Big\} + O(\epsilon^2) \Big].
\end{aligned}$$

Therefore,

$$\begin{aligned}
& {}_S \langle \phi_k | \hat{U}(\epsilon) | \phi_{k-1} \rangle_S \\
&= \int \mathfrak{D}\pi_{1,k} \mathfrak{D}\pi_{1,k}^\dagger \exp \left[i\pi_{1,k}(\phi_k - \phi_{k-1}) + i\pi_{1,k}^\dagger(\phi_k^\dagger - \phi_{k-1}^\dagger) \right. \\
&\quad \left. - \epsilon \left\{ \pi_{1,k}^\dagger \pi_{1,k} - \frac{i}{2\theta} t_2 \frac{\phi_k + \phi_{k-1}}{2} \pi_{1,k} + \frac{i}{2\theta} t_2 \frac{\phi_k^\dagger + \phi_{k-1}^\dagger}{2} \pi_{1,k}^\dagger + \frac{1}{4\theta^2} t_2^2 \phi_k^\dagger \phi_k \right. \right. \\
&\quad \left. \left. + V[\phi, \phi^\dagger; t_{1,k}] \right\} + O(\epsilon^2) \right] \\
&= \int \mathfrak{D}\pi_{1,k} \mathfrak{D}\pi_{1,k}^\dagger \exp \left[-\epsilon \pi_{1,k}^\dagger \pi_{1,k} \right. \\
&\quad \left. + i \left\{ (\phi_k - \phi_{k-1}) + \frac{\epsilon}{2\theta} t_2 \frac{\phi_k + \phi_{k-1}}{2} \right\} \pi_{1,k} + i \left\{ (\phi_k^\dagger - \phi_{k-1}^\dagger) - \frac{\epsilon}{2\theta} t_2 \frac{\phi_k^\dagger + \phi_{k-1}^\dagger}{2} \right\} \pi_{1,k}^\dagger \right. \\
&\quad \left. - \epsilon \left\{ \frac{1}{4\theta^2} t_2^2 \phi_k^\dagger \phi_k + V[\phi, \phi^\dagger; t_{1,k}] \right\} + O(\epsilon^2) \right] \\
&\quad \int dz d\bar{z} \exp \left\{ -a\bar{z}z + ibz + i\bar{b}\bar{z} \right\} \\
&\quad \sim \int dx dy \exp \left\{ -a(x^2 + y^2) + i(b + \bar{b})x - (b - \bar{b})y \right\} \\
&\quad \sim \int dx dy \exp \left\{ -a(x - i\frac{b+\bar{b}}{2a})^2 - a(y + \frac{b-\bar{b}}{2a})^2 - \frac{(b+\bar{b})^2}{4a} + \frac{(b-\bar{b})^2}{4a} \right\} \\
&\quad \sim \exp \left\{ -\frac{(b+\bar{b})^2}{4a} + \frac{(b-\bar{b})^2}{4a} \right\} \\
&\quad \sim \exp \left(-\frac{\bar{b}b}{a} \right) \\
&= \exp \left[-\frac{1}{\epsilon} \left\{ (\phi_k^\dagger - \phi_{k-1}^\dagger) - \frac{\epsilon}{2\theta} t_2 \frac{\phi_k^\dagger + \phi_{k-1}^\dagger}{2} \right\} \left\{ (\phi_k - \phi_{k-1}) + \frac{\epsilon}{2\theta} t_2 \frac{\phi_k + \phi_{k-1}}{2} \right\} \right. \\
&\quad \left. - \epsilon \left\{ \frac{1}{4\theta^2} t_2^2 \phi_k^\dagger \phi_k + V[\phi, \phi^\dagger; t_{1,k}] \right\} + O(\epsilon^2) \right] \\
&= \exp \left[-\epsilon \left\{ \left(\frac{\phi_k^\dagger - \phi_{k-1}^\dagger}{\epsilon} \right) \left(\frac{\phi_k - \phi_{k-1}}{\epsilon} \right) \right. \right. \\
&\quad \left. - \frac{1}{2\theta} t_2 \frac{\phi_k^\dagger + \phi_{k-1}^\dagger}{2} \frac{\phi_k - \phi_{k-1}}{\epsilon} + \frac{1}{2\theta} t_2 \frac{\phi_k + \phi_{k-1}}{2} \frac{\phi_k^\dagger - \phi_{k-1}^\dagger}{\epsilon} \right. \\
&\quad \left. + \frac{1}{4\theta^2} t_2^2 (\phi_k^\dagger \phi_k - \frac{\phi_k^\dagger + \phi_{k-1}^\dagger}{2} \frac{\phi_k + \phi_{k-1}}{2}) + V[\phi, \phi^\dagger; t_{1,k}] \right\} + O(\epsilon^2) \right] \\
&\equiv \exp \left[-\epsilon \left\{ * * * \right\} + O(\epsilon^2) \right].
\end{aligned}$$

Now that we have found the transition amplitude during the small interval ϵ , we can get the full amplitude :

$$\begin{aligned}
& {}_H \langle \phi_f(t_{1,f}, t_{2,f}); t_{1,f} | \phi_i(t_{1,i}, t_{2,i}); t_{1,i} \rangle_H \\
&= \lim_{\substack{\epsilon \rightarrow 0 \\ N \rightarrow \infty}} \prod_{n=1}^N \int \mathfrak{D}\phi_n \mathfrak{D}\phi_n^\dagger \prod_{k=1}^{N+1} \exp \left[-\epsilon \left\{ *** \right\} \right] \\
&= \lim_{\substack{\epsilon \rightarrow 0 \\ N \rightarrow \infty}} \prod_{n=1}^N \int \mathfrak{D}\phi_n \mathfrak{D}\phi_n^\dagger \exp \left[-\epsilon \sum_{k=1}^{N+1} \left\{ *** \right\} \right] \\
&= \int \mathfrak{D}\phi \mathfrak{D}\phi^\dagger \exp \left[- \int_{t_{1,i}}^{t_{1,f}} dt_1 \int_{t_{2,i}}^{t_{2,f}} dt_2 \left\{ \partial_1 \phi^\dagger \partial_1 \phi - \frac{1}{2\theta} (\phi^\dagger t_2 \partial_1 \phi - \phi t_2 \partial_1 \phi^\dagger) \right. \right. \\
&\quad \left. \left. + \partial_2 \phi^\dagger \partial_2 \phi + \frac{1}{4\theta^2} (t_1^2 - t_2^2) \phi^\dagger \phi + \frac{i}{2\theta} (\phi^\dagger t_1 \partial_2 \phi - \phi t_1 \partial_2 \phi^\dagger) + m^2 \phi^\dagger \phi \right\} \right] \\
&= \int \mathfrak{D}\phi \mathfrak{D}\phi^\dagger \exp \left[- \int_{t_{1,i}}^{t_{1,f}} dt_1 \int_{t_{2,i}}^{t_{2,f}} dt_2 \left\{ \partial_1 \phi^\dagger \partial_1 \phi + \partial_2 \phi^\dagger \partial_2 \phi \right. \right. \\
&\quad \left. \left. + \frac{1}{4\theta^2} (t_1^2 + t_2^2) \phi^\dagger \phi + \frac{i}{2\theta} (\phi^\dagger t_1 \partial_2 \phi - \phi^\dagger t_2 \partial_1 \phi - \phi t_1 \partial_2 \phi^\dagger + \phi t_2 \partial_1 \phi^\dagger) + m^2 \phi^\dagger \phi \right. \right. \\
&\quad \left. \left. - \frac{1}{2\theta^2} t_2^2 \phi^\dagger \phi + (1+i) \frac{i}{2\theta} (\phi^\dagger t_2 \partial_1 \phi - \phi t_2 \partial_1 \phi^\dagger) \right\} \right]. \tag{65}
\end{aligned}$$

This is the functional integral of the (free) scalar field theory on noncommutative space. Note that the exponential part is of the form $\exp[-S + (\text{extra terms})]$ (S be the “Euclidean” action (29)) because of the term $\ll \text{angular momentum} \gg$ and the strange commutation relations (47)(48).

According to the usual method, in order to introduce the interaction, let us define the generating functional $Z_0[J, J^\dagger]$ for free theory.

$$\begin{aligned}
Z_0[J, J^\dagger] \equiv \int \mathfrak{D}\phi \mathfrak{D}\phi^\dagger \exp \left[- \int dt^2 \phi^\dagger \left\{ \frac{1}{\theta} (\hat{N} + 1) + \frac{1}{\theta} \hat{L} + m^2 + \hat{M} \right\} \phi \right. \\
\left. + \int dt^2 (J\phi + J^\dagger \phi^\dagger) \right] \tag{66}
\end{aligned}$$

$$\hat{M} \equiv -\frac{1}{2\theta^2} t_2^2 + (1+i) \frac{i}{\theta} t_2 \partial_1 \tag{67}$$

$Z_0[J, J^\dagger]$ is supposed to be normalized such that $Z_0[J = 0, J^\dagger = 0] = 1$. The term $\left[- \int dt^2 \phi^\dagger \hat{M} \phi \right]$ is the (extra terms) mentioned above. Before turning to a explicit evaluation of $Z_0[J, J^\dagger]$, the operator \hat{M} must be clarified. As we shall see in the following calculation, the operator \hat{M} can be expressed with the “raising and lowering” operators $\hat{N}_\pm, \hat{N}_\pm^\dagger$. From (34)(35), we obtain

$$\begin{aligned}
t_2 &= \frac{i}{2} \theta \{ (\hat{N}_+ + \hat{N}_-) - (\hat{N}_+^\dagger + \hat{N}_-^\dagger) \} \\
\partial_1 &= \frac{1}{4} \{ (\hat{N}_+ + \hat{N}_-) - (\hat{N}_+^\dagger + \hat{N}_-^\dagger) \}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{M} &= -\frac{1}{2\theta^2}t_2^2 + (1+i)\frac{i}{\theta}t_2\partial_1 \\
&= \frac{1}{8}\{(\hat{N}_+ + \hat{N}_-) - (\hat{N}_+^\dagger + \hat{N}_-^\dagger)\}^2 - \frac{1}{8}(1+i)\{(\hat{N}_+ + \hat{N}_-)^2 - (\hat{N}_+^\dagger + \hat{N}_-^\dagger)^2\} \\
&= \frac{1}{8}[2\{(\hat{N}_+^\dagger + \hat{N}_-^\dagger)^2 - (\hat{N}_+ + \hat{N}_-)(\hat{N}_+^\dagger + \hat{N}_-^\dagger)\} - i\{(\hat{N}_+ + \hat{N}_-)^2 - (\hat{N}_+^\dagger + \hat{N}_-^\dagger)^2\}], \\
\langle N, l | \hat{M} | N', l' \rangle &= \langle N, l | \{ \frac{1}{4}(\hat{N}_+^\dagger \hat{N}_-^\dagger + \hat{N}_-^\dagger \hat{N}_+^\dagger) + \frac{i}{8}(\hat{N}_+^\dagger \hat{N}_-^\dagger + \hat{N}_-^\dagger \hat{N}_+^\dagger) - \frac{i}{8}(\hat{N}_+ \hat{N}_- + \hat{N}_- \hat{N}_+) \} | N', l' \rangle \\
&\quad + (\text{remainders}) \\
&= -\frac{1}{2}\{\frac{1}{\theta}(N+1) + \frac{1}{\theta}l\}\delta_{NN'}\delta_{ll'} - \frac{i}{2}\frac{1}{\theta}l\delta_{NN'}\delta_{ll'} + (\text{remainders}). \tag{68}
\end{aligned}$$

Here, (remainders) are the terms which are not proportional to $\delta_{NN'}\delta_{ll'}$, such as

$$\langle N, l | \hat{N}_+^\dagger \hat{N}_+^\dagger | N', l' \rangle \propto \delta_{N, N'+2} \delta_{l, l'+2}.$$

Now that we have got the necessary information, let us evaluate the generating functional $Z_0[J, J^\dagger]$. From this evaluation, we will obtain the propagator, and be able to introduce the multi-point “local” interaction. Although the practical calculation is slightly tedious, we shall show it in detail.

$$\begin{aligned}
Z_0[J, J^\dagger] &= \int \mathfrak{D}\phi \mathfrak{D}\phi^\dagger \exp \left[- \int dt^2 \phi^\dagger \left\{ \frac{1}{\theta} (\hat{N} + 1) + \frac{1}{\theta} \hat{L} + m^2 + \hat{M} \right\} \phi + \int dt^2 (J\phi + J^\dagger \phi^\dagger) \right] \\
&\quad \left\{ \begin{aligned} \phi(t_1, t_2) &= \sum_{N,l} \alpha_{N,l} \langle t_1, t_2 | N, l \rangle \\ \phi^\dagger(t_1, t_2) &= \sum_{N,l} \alpha_{N,l}^\dagger \langle N, l | t_1, t_2 \rangle \\ J(t_1, t_2) &= \sum_{N,l} J_{N,l} \langle t_1, t_2 | N, l \rangle \\ J^\dagger(t_1, t_2) &= \sum_{N,l} J_{N,l}^\dagger \langle t_1, t_2 | N, l \rangle \end{aligned} \right. \\
&= \prod_{N=0}^{\infty} \prod_{-N \leq l \leq N} \int d\alpha_{N,l} d\alpha_{N,l}^\dagger \\
&\quad \times \exp \left[- \sum_{N=0}^{\infty} \sum_{-N \leq l \leq N} \left\{ \frac{1}{2\theta} (N+1) + \frac{1}{2\theta} l + m^2 - \frac{i}{2\theta} l \right\} \alpha_{N,l}^\dagger \alpha_{N,l} \right. \\
&\quad \left. + (\text{remainders}) + \sum_{N=0}^{\infty} \sum_{-N \leq l \leq N} (J_{N,l} \alpha_{N,l} + J_{N,l}^\dagger \alpha_{N,l}^\dagger) \right] \\
&\quad \left(\begin{aligned} &\text{Let us write the (remainders)' contribution symbolically.} \\ &\int d\alpha_N d\alpha_N^\dagger \exp[-(\alpha_N^\dagger \alpha_N + \alpha_N^\dagger \alpha_{N+2})], \quad \alpha_N \equiv x_N + iy_N \\ &\sim \int dx_N dy_N \exp[-(x_N^2 + y_N^2) \\ &\quad + (x_{N+2} + iy_{N+2})x_N - i(x_{N+2} + iy_{N+2})y_N] \\ &\sim \exp\left[\frac{(x_{N+2} + iy_{N+2})^2}{4} - \frac{(x_{N+2} + iy_{N+2})^2}{4}\right] \\ &\sim 1 \end{aligned} \right) \\
&= \exp \left[\sum_{N=0}^{\infty} \sum_{-N \leq l \leq N} J_{N,l}^\dagger \left\{ \frac{1}{2\theta} (N+1) + \frac{1}{2\theta} l + m^2 - \frac{i}{2\theta} l \right\}^{-1} J_{N,l} \right] \tag{69}
\end{aligned}$$

From this, we find that the propagator in “momentum” space (N, l) is

$$\left\{ \frac{1}{2\theta} (N+1) + \frac{1}{2\theta} l + m^2 - \frac{i}{2\theta} l \right\}^{-1}. \tag{70}$$

The generating functional $Z[J, J^\dagger]$ for interacting theory can be written in the same way.

$$\begin{aligned}
Z[J, J^\dagger] &= \int \mathfrak{D}\phi \mathfrak{D}\phi^\dagger \exp \left[-S - \int dt^2 \phi^\dagger \hat{M} \phi - \int dt^2 \mathfrak{L}_{\text{int}}[\phi, \phi^\dagger] + \int dt^2 (J\phi + J^\dagger \phi^\dagger) \right] \\
&\quad \left(\text{for example, } \mathfrak{L}_{\text{int}} = \frac{\lambda}{4} (\phi^\dagger)^2 (\phi)^2 \right) \tag{71}
\end{aligned}$$

The Feynman rule is much the same as the usual interacting scalar field theory, except giving the lines the propagator (70) and taking summation (N, l) for the inner lines.

As is well known, the usual scalar field theory has the logarithmic UV divergences, but I do not know about (71) enough to be able to say that it's amplitudes diverge or converge, yet. As the spectrum (N, l) is discrete, however, the degree of divergence is expected to be lowered, or the existence of the imaginary part in the propagator (70) may make the loop amplitudes convergent.

As the last comment of this section, let us refer to the “Minkovskian” functional integral. Starting from the action (52) and using the “real-time” evolution operator, we will arrive at the functional integral of the integrand $\exp[-S_M]$ after a similar calculation. This is, in a sense, the preferable result. But, instead of the strange functional integral in the “Euclidean” case, it is difficult to find the complete system which expands ϕ and ϕ^\dagger in the action S_M , in other words, difficult to first-quantize the system S_M . In future, I would like to examine the action S_M closely.

6. Discussions and acknowledgements

In this paper, we constructed the model of the scalar field theory on the noncommutative space from the point of view that the first-quantization is equal to making the space-time noncommutative and that the first-quantized quantum mechanics is equal to the classical field theory. We carried out the first, second quantizations. The advantage of our formulation is that the method is very similar to that of the usual scalar field theories contrary to the approach of using the \ast -product. The latter has the logarithmic UV divergences and also the IR ones. Our main results are summarized in the following two. One is that the spectrum is discrete given by (N, l) . The other is that the propagator (70) has the imaginary part which looks like the regulator. Thus we may expect the loop amplitudes convergent. But investigation of the symmetries and conserved quantities are left to the future works. As our action S in (29) manifestly depends on the space coordinates t_1, t_2 , it doesn't keep the symmetry of translation. Therefore, the usual energy-momentum is not conserved. But the action S maintains the rotational invariance in the space (t_1, t_2) , and the relation between this rotational symmetry and the real Lorentz invariance should be clarified.

Although, for simplicity, we have used the “Euclidean” noncommutative space, and got the “Euclidean” functional integral with the “Minkovskian” system which evolves in the “imaginary-time”, we can discuss in the “Minkovskian” metric instead of “Euclidean” equation (12). In this case, the first-quantization cannot be done by the harmonic oscillator system, and one cannot say something definite about the spectrum. The functional integral by the evolution in the “real” time, however, becomes the familiar form such as $\int \mathcal{D}\phi \exp[-S_M]$. So, if we can find the complete system of the first-quantized theory, it may be easy to treat the functional integral.

As a interesting extension, we can also impose the noncommutativity on the fields ϕ

and ϕ^\dagger such as

$$\begin{aligned} [\phi(t_1, t_2), \phi(t_1, t'_2)] &= i\theta(t_2 - t'_2) \\ [\phi^\dagger(t_1, t_2), \phi^\dagger(t_1, t'_2)] &= i\theta(t_2 - t'_2). \\ \theta(x) &= \begin{cases} +\frac{1}{2} & , x > 0 \\ 0 & , x = 0 \\ -\frac{1}{2} & , x < 0 \end{cases} \end{aligned}$$

This noncommutativity will lead us to the strange “creation-annihilation” algebra and functional integral which also fairly differ from (47)(48) and (65), and produce another effects on the convergence of amplitudes.

As the other future works, the introduction of the gauge interactions may be interesting. As is well known (see, for example, Ref. [3]), the gauge potential and the connection can be introduce into the context of the noncommutative geometry. From the point of view that the first-quantization is by itself the construction of the field theories, our new method may be needed for the noncommutative version of Yang-Mills theory and gravity model. At least, the degree of divergence seems to be lowered in our model, so the application of our method to the other field theories may be useful.

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